Dynamic Lyapunov Indicator: a practical tool for distinguishing between ordered and chaotic orbits in discrete dynamical systems

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Abstract: - In the present paper our intention was to verify if the Dynamic Lyapunov Indicator (DLI), proposed recently by Saha and Budhraja as a new tool for distinguishing between ordered and chaotic orbits, gives correct conclusions when is applied to discrete dynamical systems. With this end in view, the behavior of the 2-D Lozi map, the 2-D predator prey map and the 3-D Lorentz BD map is analyzed with the help of DLI and, for comparison, with the help of Fast Lyapunov Indicator (FLI), a consecrated tool in chaos theory. Results obtained seem to qualify the DLI as a new indicator of chaos, at least for discrete dynamical systems.

Key-Words: - Indicator of chaos, Dynamic Lyapunov Indicator, maps

1 Introduction

The study of chaos is currently very popular, but the phenomenon is not new and has been of interest to astronomers and mathematicians for about one hundred years. Much of the interest is due to the case of the computer as a tool for making empirical observations. Today, chaos theory is applied in many scientific disciplines: mathematics, biology, computer science, economics, engineering, finance, philosophy, physics, politics, population dynamics, psychology and robotics.

Researchers in the field of chaos have always tried to find new methods to characterize chaos and may efficiently differentiate between chaos and regular motion in discrete and continuous dynamical systems. In the past there were certain tools to identify regular and chaotic orbits such as time series curves, phase plots, Poincare maps, power spectra, Lyapunov exponents etc. Most recently, new tools have been discovered, like Fast Lyapunov Indicator (FLI), smaller Alignment Indices (SALI), 0-1 test etc. In a recent paper [2],Saha and Budhraja studied a new indicator and applied it on the Henon map and on the Standard map. They specified very clear that other verifications are necessary before accepting this indicator as a tool of chaotic and regular motion.

The main goal of our work is to do this on other discrete dynamical systems. The paper is organized as follows. In Section 2 we recall the definitions of the FLI and of the new indicator. In Section 3 we compute these indicators on some regular and chaotic orbits of three discrete maps. Conclusions and future developments are provided in Section 4.

2 The FLI and the new indicator

2.1 The Fast Lyapunov Indicator (FLI)

FLI was introduced by Froeschle et al (1997) as a means of detecting chaotic and regular motions [1]. The FLI is defined as follows:

Starting with a *m*-dimensional basis

$$V_m(0) = \left(v_1(0), v_2(0), \ldots, v_m(0)\right)$$

embedded in an *n*-dimensional space with an initial condition $\left(x_1(0), x_2(0), \ldots, x_n(0)\right)$, we take at each iteration the largest amongst the vectors of the evolving basis. Thus, the FLI is defines as:

$$FLI = \sup\left\|v_j\right\|,\ j = 1,2,..,m \tag{1}$$

Froeschle has shown that FLI increases exponentially for chaotic orbits and linearly for regular orbits.

2.2. The New Indicator

Saha and Budhraja are defined a new indicator, called Dynamic Lyapunov Indicator (DLI), as follows:

Let J be the Jacobian matrix of the discrete dynamical system (map). At every discrete time, we calculate the eigenvalues of the matrix J and then plot the largest eigenvalue. It seems that these

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eigenvalues form a definite pattern for regular motion and are distributed randomly for chaotic orbits.

3 Applications of FLI and DLI for certain maps

We have applied above defined indicators for the models given below:

3.1. Lozi map

It is a two-dimensional map introduced by Lozi(1978):

$$x_{n+1} = 1 - a|x_n| + b\, y_n \;,\; y_{n+1} = x_n \qquad (2)$$

where a and b are non-zero parameters. This map evolves chaotically when $a = 1.7, b = 0.5$ and initial conditions $(x_0, y_0) = (-0.1, 0.1)$. Indeed, Lyapunov exponents are $\lambda_1 = 0.47023$ and $\lambda_2 = -1.16338$ (see Fig. 1 and Fig. 2).

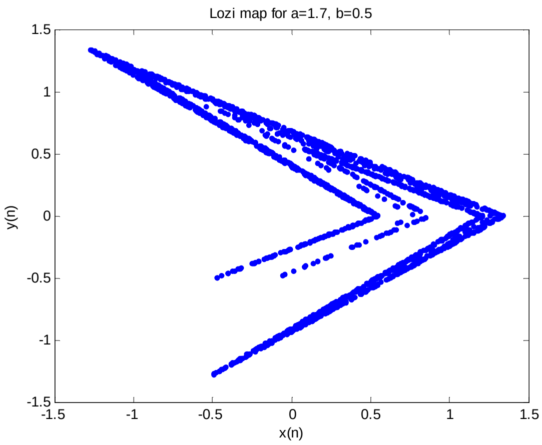

Fig.1: Phase plot of Lozi map for $a = 1.7, b = 0.5$

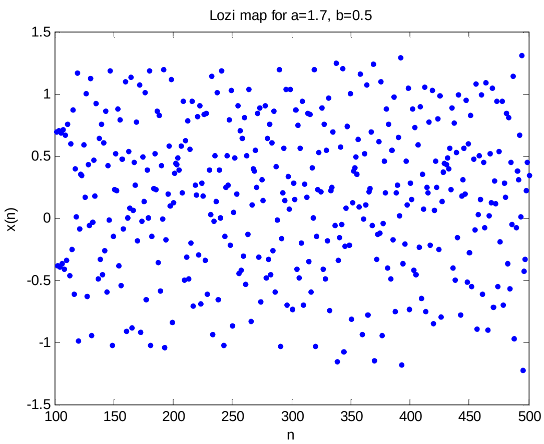

Fig.2: *x(n)* versus *n* plot for Lozi map with $a = 1.7, b = 0.5$

The exponential increase of FLI in Fig. 3 indicates the orbit is chaotic. The ordinate is taken with base 10.

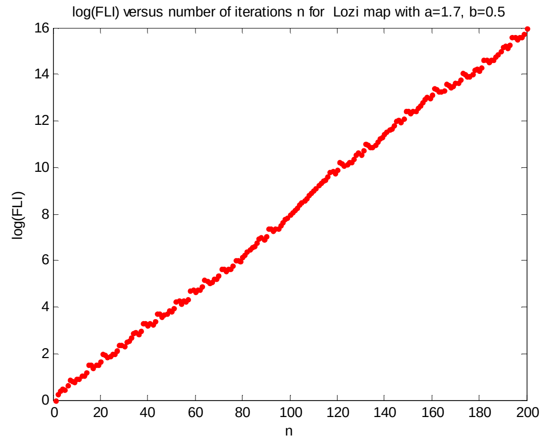

Fig. 3: Log(FLI) plots for chaotic Lozi Map

A random distribution for the largest eigenvalue is obvious from Fig. 4.

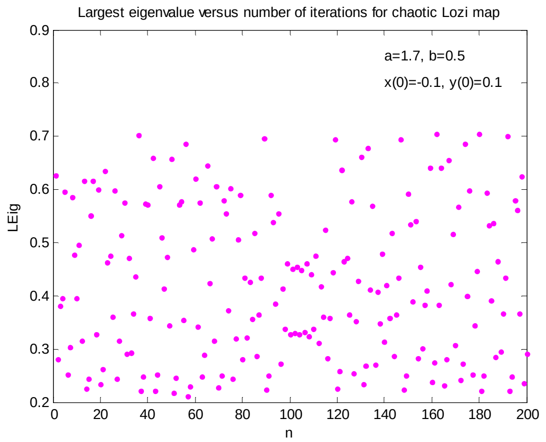

Fig.4: DLI plots for chaotic Lozi map

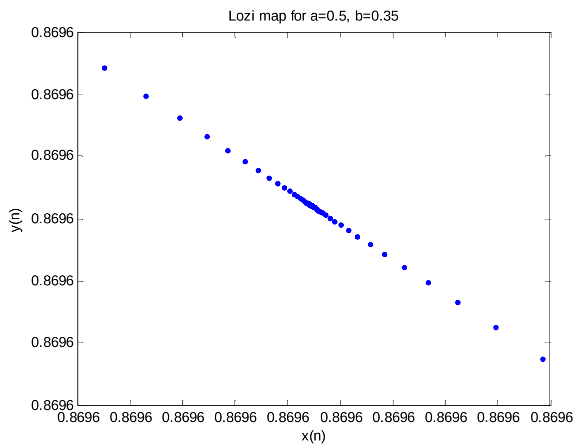

Fig. 5: Phase plot of Lozi map for $a = 0.5, b = 0.35$

The Lozi map evolves to the fixed point $(0.869832, 0.869832)$ for $a = 0.5, b = 0.35$ and

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initial conditions $(x_0, y_0) = (0.25, 0.1)$ (see Fig. 5 and Fig. 6).

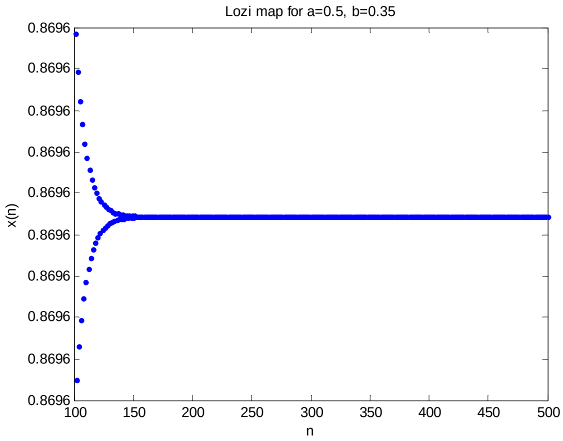

Fig.6: *x(n)* versus *n* plot for Lozi map with $a = 0.5, b = 0.35$

Log(FLI) tends to zero and a pattern in DLI plots is clearly visible for this case (see Fig. 7 and Fig. 8).

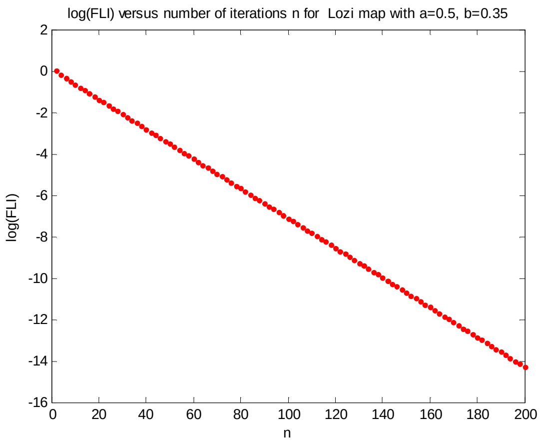

Fig. 7: Log(FLI) plots for Lozi Map with $a = 0.5, b = 0.35$

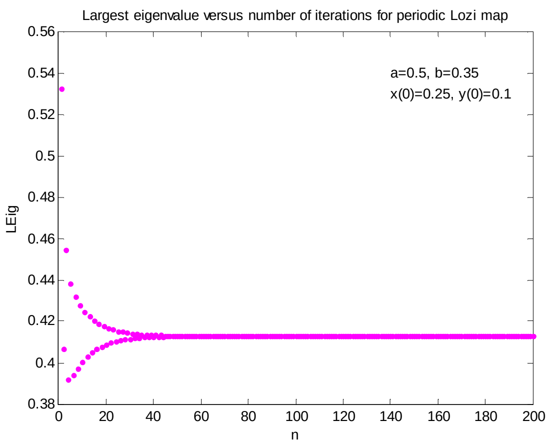

Fig.8: DLI plots for Lozi map with $a = 0.5, b = 0.35$

The same types of results are displayed through the Figs. 9-12 for $a = b = 0.5$, $(x_0, y_0) = (0.25, 0.1)$, who corresponds to a 2-period orbit.

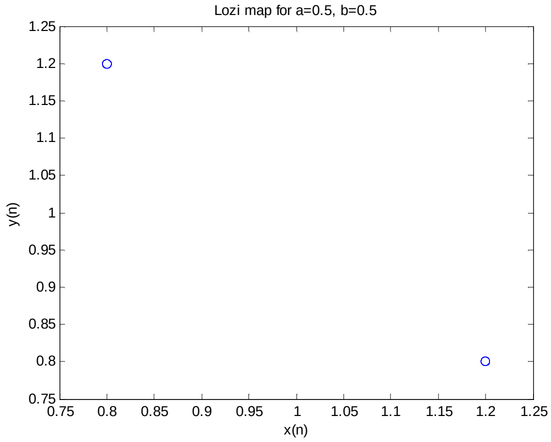

Fig. 9: Phase plot of Lozi map for $a = b = 0.5$

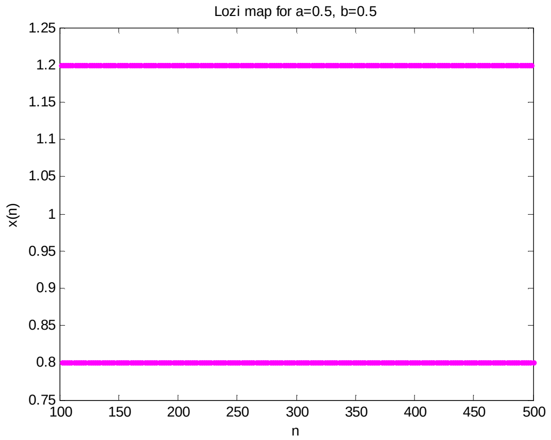

Fig.10: *x(n)* versus *n* plot for Lozi map ($a = b = 0.5$)

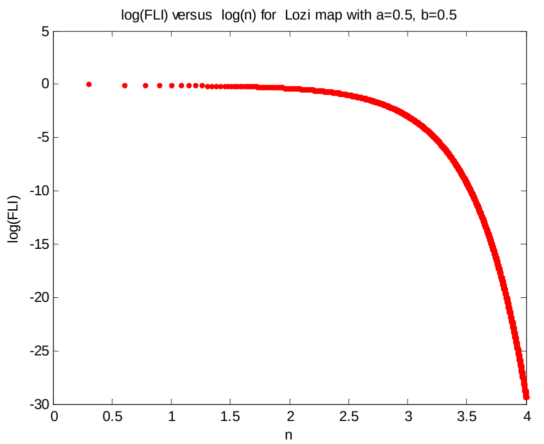

Fig. 11: Log(FLI) plots for Lozi Map with $a = b = 0.5$

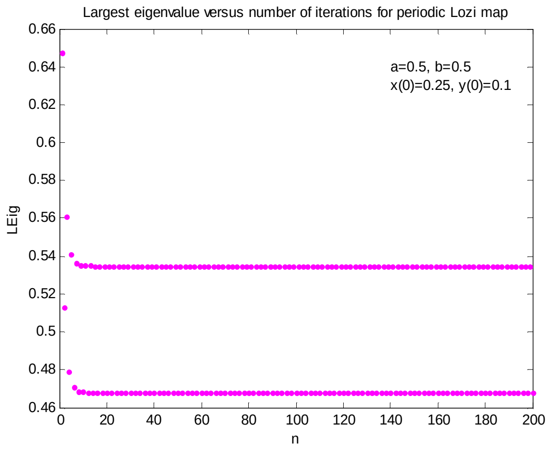

Fig.12: DLI plots for Lozi map with $a = b = 0.5$

3.2. Discrete predator-prey map

The 2-D predator-prey map is described by the

$$
\begin{aligned}
x_{n+1} &= x_n \exp[r(1 - x_n / k) - a y_n] \\
y_{n+1} &= x_n [1 - \exp(-a y_n)]
\end{aligned}
\tag{2}
$$

with *r*, *k* and *a* real constants.

For $r = 3, k = 1, a = 5$ and $(x_0, y_0) = (0.5, 0.5)$ the Lyapunov exponents are $\lambda_1 = 0.19664, \lambda_2 = 0.03276$ and indicate chaotic orbit (see Fig. 13).

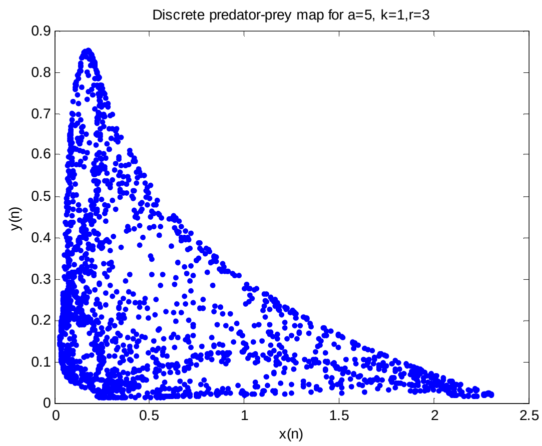

Fig. 13: Phase plot of predator-prey map for $r = 3, k = 1, a = 5$

FLI increase exponentially to about 10^{45} in just 300 iterations and DLI has a random behavior (see Figs. 14 and 15).

For the ordered orbit presented in Fig. 16, for which $r = 1.2, k = 2.42, a = 5$, the behavior of log(FLI) is somehow similar to that shown in chaotic case but in the first 1000 iterations the growth of FLI is almost linear (the value $FLI \cong 10^{45}$ is obtained after 7000 iterations).

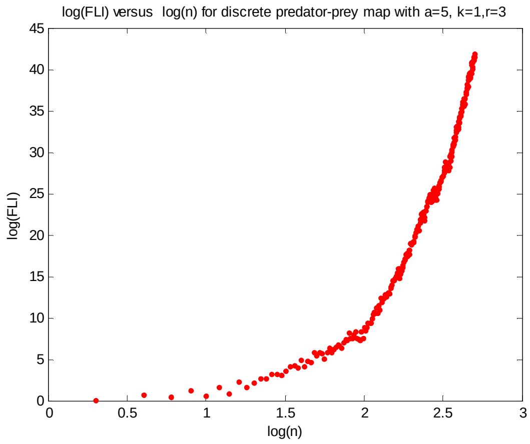

Fig. 14: Log(FLI) plots for predator-prey map with $r = 3, k = 1, a = 5$

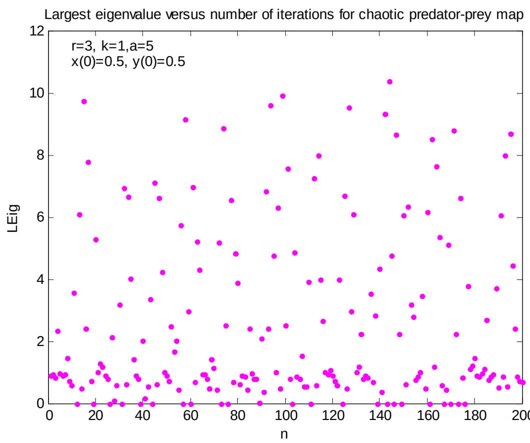

Fig.15: DLI plots for predator-prey map with $r = 3, k = 1, a = 5$

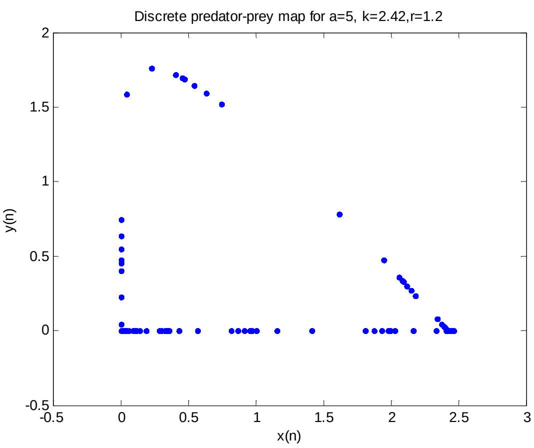

Fig. 16: Phase plot of predator-prey map for $r = 1.2, k = 2.42, a = 5$

A pattern is clearly visible in DLI plots (see Fig. 18). In the end, for a 12-period orbit ($r = 1.28$,

$k = 2.42, a = 5$) log(FLI) decreases to about 10^{-70} in 1000 iterations while DLI, after a transition period, describes a right line (see Figs. 19-21).

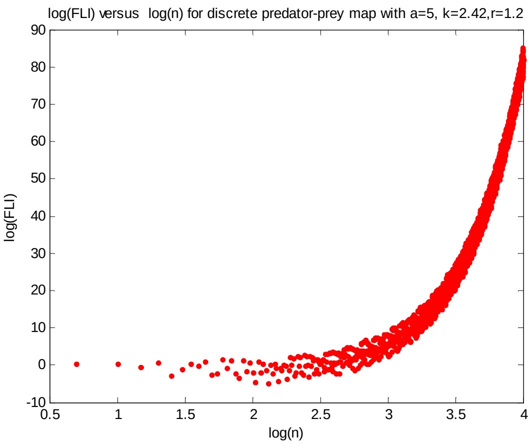

Fig. 17: Log(FLI) plots for predator-prey map with $r = 1.2, k = 2.42, a = 5$

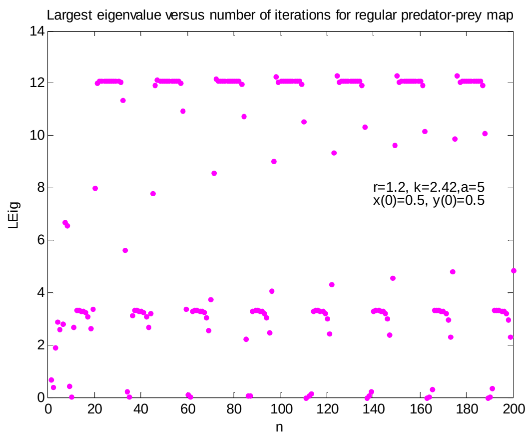

Fig.18: DLI plots for predator-prey map with $r = 1.2, k = 2.42, a = 5$

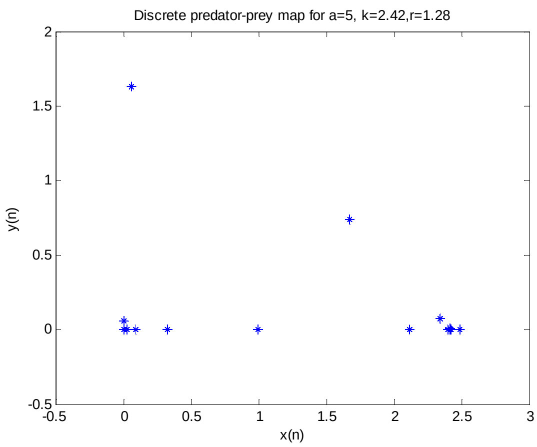

Fig. 19: Phase plot of predator-prey map for $r = 1.28, \quad k = 2.42, a = 5$

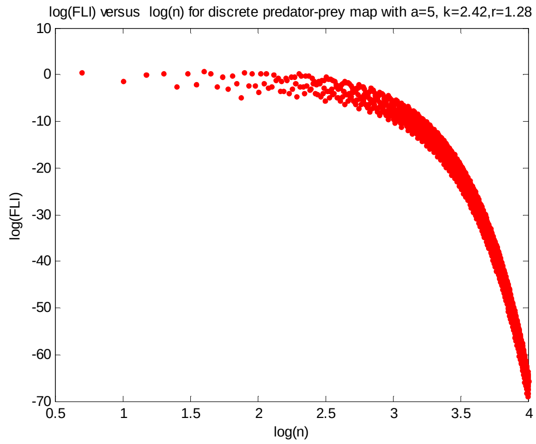

Fig. 20: Log(FLI) plots for predator-prey map with $r = 1.28, \quad k = 2.42, a = 5$

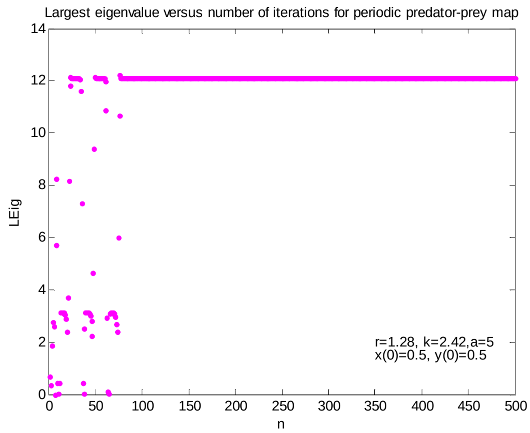

Fig.21: DLI plots for predator-prey map with $r = 1.28, \quad k = 2.42, a = 5$

3.3. Lorentz BD map

As the last example we get the 3-D Lorentz map

$$x_{n+1} = x_n \, y_n - z_n, \; y_{n+1} = x_n, \; z_{n+1} = y_n \qquad (3)$$

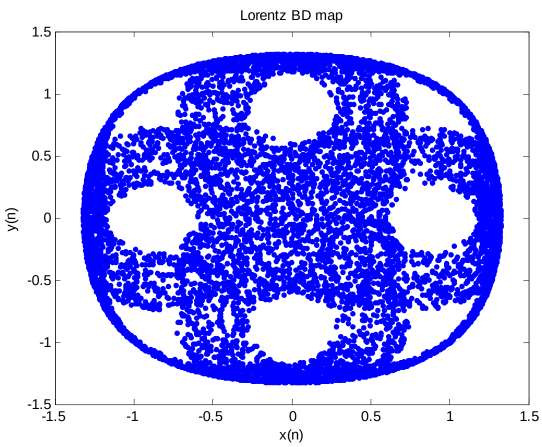

Fig. 22: Phase plot of Lorentz BD map for $(x_0, y_0, z_0) = (0.5, 0.5, -1)$

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This map evolves chaotically when $(x_0, y_0, z_0) = (0.5, 0.5, -1)$ and it is regular when $(x_0, y_0, z_0) = (0.5, 0.5, 1)$ (See Figs. 22 and 23).

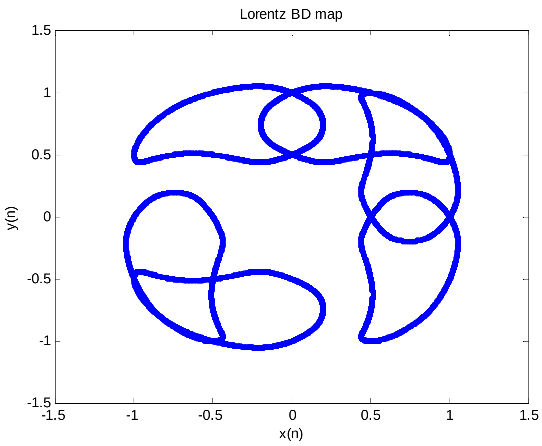

Fig. 23: Phase plot of Lorentz BD map for $(x_0, y_0, z_0) = (0.5, 0.5, 1)$

FLI grow exponentially for the chaotic orbit (to about 10^{160} in 3000 iterations) and show a linear behavior for the periodic orbit (see Figs. 24 and 25).

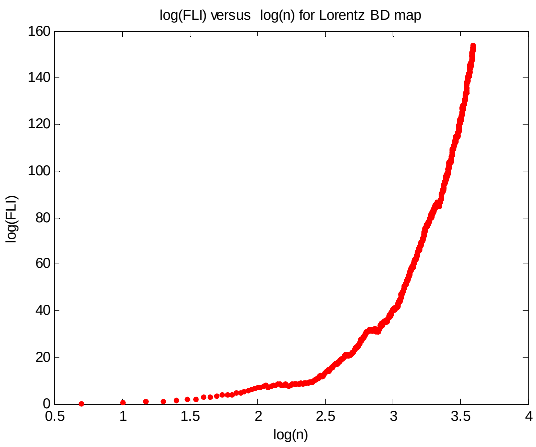

Fig. 24: Log(FLI) plots for the chaotic Lorentz map

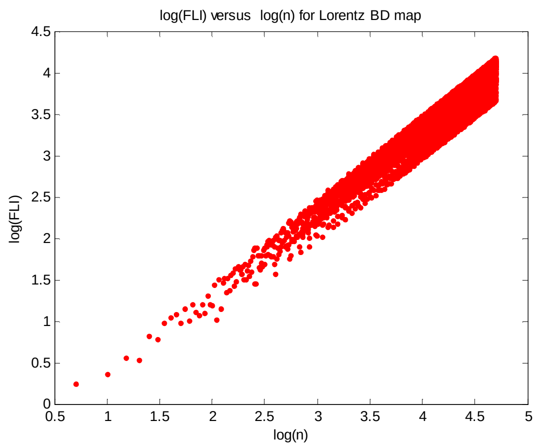

Fig. 25: Log(FLI) plots for the ordered Lorentz map

As Saha and Buhraja said, the DLI plots form a definite pattern in regular case (Fig. 26) and are distributed randomly in chaotic case.

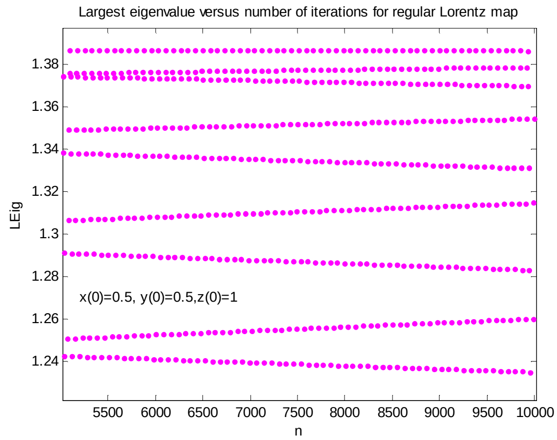

Fig.26: DLI plots for chaotic Lorentz map

4 Conclusion

The aim of this paper it was to apply the FLI and DLI methods for distinguishing between ordered and chaotic orbits in the case of some discrete-time dynamical systems. We investigated the 2-D Lozi map, the 2-D predator-prey map and the 3-D Lorentz BD map. The main conclusions of our study are:

a) The FLI increases exponentially for chaotic orbits and decreases to zero or presents a linear variation for a regular orbit;

b) The DLI behaves randomly for chaotic orbits and regularly for ordered orbits;

c) The two indicators give very clear indications about orbit's nature whenever applied; Their computation is fast and easy; Only few hundreds of iterations are sufficient to get a conclusion;

d) Before accepting DLI as an indicator of chaotic and regular motion, other studies are necessary, especially for continuous dynamical systems.

References:

[1] Lega,E., Froeschle, C., On the Relationship between Fast Lyapunov Indicator and Periodic Orbits for Symplectic Mappings, *Celestial Mechanics and Dynamical Astronomy*, Vol.81, 2001, pp. 129-147.

[2] Saha, L.M., Budhraja,M., The Largest Eigenvalue: An Indicator of Chaos?, Int.J. of Appl.Math and Mech, Vol 3(1), 2007, pp. 61-71.